

SOME ASPECTS OF NONCOMMUTATIVE GEOMETRY AND PHYSICS

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Abstract

An introduction is given to some selected aspects of noncommutative geometry. Simple examples in this context are provided by finite sets and lattices. As an application, it is explained how the nonlinear Toda lattice and a discrete time version of it can be understood as generalized σ -models based on noncommutative geometries. In particular, in this way one achieves a simple understanding of the complete integrability of the Toda lattice. Furthermore, generalized metric structures on finite sets and lattices are briefly discussed.

1 Introduction

Noncommutative (differential) geometry extends notions of classical differential geometry from differentiable manifolds to discrete spaces, like finite sets and fractals, and even ‘non-commutative spaces’ (‘quantum spaces’) which are given by noncommutative associative algebras (over \mathbb{R} or \mathbb{C}). Such an algebra \mathcal{A} replaces the commutative algebra of C^∞ -functions on a smooth manifold \mathcal{M} .

A basic concept of classical differential geometry is the notion of a vector field. The latter is a derivation $C^\infty(\mathcal{M}) \rightarrow C^\infty(\mathcal{M})$. One may think of generalizing the notion of a vector field as a derivation $\mathcal{A} \rightarrow \mathcal{A}$ to associative algebras.[1] But there are interesting algebras, like the (commutative) algebra of functions on a finite set, on which there is no derivation at all except the trivial one (which maps all functions to zero). It seems that there is no adequate general definition of a vector field.

Perhaps of even more importance than vector fields are differential forms in classical differential geometry and theoretical physics. They extend the algebra $\Omega^0(\mathcal{M}) := C^\infty(\mathcal{M})$ to a graded associative algebra $\Omega(\mathcal{M}) = \bigoplus_{r \geq 0} \Omega^r(\mathcal{M})$, the algebra of differential forms. A central part of its structure is encoded in the action of the exterior derivative $d : \Omega^r(\mathcal{M}) \rightarrow \Omega^{r+1}(\mathcal{M})$ which is a (graded) derivation. Given any associative algebra \mathcal{A} , one can always associate with it a differential algebra $\Omega(\mathcal{A})$ which should be regarded as an analogue of the algebra of classical differential forms on a manifold. Further geometric structures, like connections and tensors, can then be built on such a differential algebra in very much the same way as in classical differential geometry.

There is no universal way of associating a unique differential algebra with a given associative algebra \mathcal{A} . In general, there is no distinguished differential algebra and one has to understand what the significance is of the different choices. Even in the case of the algebra of smooth functions on a manifold, there is actually no longer a good argument to single out the ordinary calculus of differential forms. The latter is distinguished, however, via our classical conception how to measure volumes. The exploration of other differential calculi opened a door to a new world of geometry and applications in physics, as well as relations with other fields of mathematics. Some aspects have been briefly reviewed in a recent paper [2] to which we refer for further information.¹

The formalism of noncommutative geometry is an extremely radical abstraction of ordinary differential geometry. It includes the latter as a special case, but allows a huge variety of different structures. In particular, it is possible to ‘deform’ ordinary differential geometry and models built on it while keeping basic concepts and recipes on which the models are based.

In the following we review some aspects of noncommutative geometry concentrating on a few easily accessible examples. Section 2 collects basic definitions of differential calculus on associative algebras. Section 3 recalls some facts about differential calculus on finite sets and a correspondence between first order differential calculi and digraphs.[3] Also a relation with an approach of Alain Connes [4] to noncommutative geometry is explained. Section 4 treats differential calculus on linear and hypercubic lattices. The corresponding differential calculi may be regarded as deformations of the ordinary differential calculus on \mathbb{R}^n . A more general class of such differential calculi is briefly discussed in section 5. The ‘lattice differential calculus’ underlies an important example of a discrete ‘generalized σ -model’. As explained in section 6, such models generalize a class of completely integrable two-dimensional classical σ -models [5] by replacing the ordinary differential calculus by a noncommutative one.[6, 7] They involve a generalized Hodge \star -operator. In classical (Riemannian) differential geometry, the Hodge operator is obtained from a metric which defines the distance between two points of the manifold. Distance functions on a rather general class of algebras have been introduced by Connes.[4] In particular, his definition applies to discrete sets. But the relation with a Hodge operator has still to be worked out. Section 7 is devoted to a discussion of some metric aspects. Section 8 contains some final remarks.

2 Differential calculi on associative algebras

Let \mathcal{A} be an associative algebra over \mathbb{R} or \mathbb{C} with unit $\mathbf{1}$. A *differential calculus* on \mathcal{A} is a \mathbb{Z} -graded associative algebra (over \mathbb{R} , respectively \mathbb{C})²

$$\Omega(\mathcal{A}) = \bigoplus_{r \geq 0} \Omega^r(\mathcal{A}) \quad (1)$$

where the spaces $\Omega^r(\mathcal{A})$ are \mathcal{A} -bimodules³ and $\Omega^0(\mathcal{A}) = \mathcal{A}$. There is a (\mathbb{R} - respectively \mathbb{C} -) linear map

$$d : \Omega^r(\mathcal{A}) \rightarrow \Omega^{r+1}(\mathcal{A}) \quad (2)$$

with the following properties,

$$d^2 = 0 \quad (3)$$

¹At present, this can be accessed online via <http://kaluza.physik.uni-konstanz.de/2MS>.

²Though in many interesting cases one has $\Omega^r(\mathcal{A}) = \{0\}$ when r is larger than some $r_0 \geq 0$, one encounters examples where $\Omega(\mathcal{A})$ is actually an infinite sum. $\Omega(\mathcal{A})$ is then the space of all finite sums of arbitrary order.

³The elements of $\Omega^r(\mathcal{A})$, called *r-forms*, can be multiplied from left and right by elements of \mathcal{A} .

$$d(w w') = (dw) w' + (-1)^r w dw' \quad (4)$$

where $w \in \Omega^r(\mathcal{A})$ and $w' \in \Omega(\mathcal{A})$. The last relation is known as the (generalized) *Leibniz rule*. One also requires $\mathbb{I} w = w \mathbb{I} = w$ for all elements $w \in \Omega(\mathcal{A})$. The identity $\mathbb{I} \mathbb{I} = \mathbb{I}$ then implies

$$d\mathbb{I} = 0. \quad (5)$$

We assume that d generates the spaces $\Omega^r(\mathcal{A})$ for $r > 0$ in the sense that $\Omega^r(\mathcal{A}) = \mathcal{A} d\Omega^{r-1}(\mathcal{A}) \mathcal{A}$. Using the Leibniz rule, every element of $\Omega^r(\mathcal{A})$ can be written as a linear combination of monomials $a_0 da_1 \cdots da_r$. The action of d is then determined by

$$d(a_0 da_1 \cdots da_r) = da_0 da_1 \cdots da_r. \quad (6)$$

So far nothing has been said about commutation relations for elements of \mathcal{A} and differentials. Indeed, in the largest differential calculus, the *universal differential envelope* $(\tilde{\Omega}(\mathcal{A}), \tilde{d})$ of \mathcal{A} , there are no such relations. Smaller differential calculi are obtained by specifying corresponding commutation relations (which have to be consistent with the existing relations in the differential algebra, of course). The smallest differential calculus is $\Omega(\mathcal{A}) = \mathcal{A}$ where d maps all elements of \mathcal{A} to zero.

3 Differential calculi on a finite set

Let \mathcal{M} be a finite set and \mathcal{A} the algebra of all \mathbb{C} -valued functions on it. \mathcal{A} is generated by $\{e_i\}$ where $e_i(j) = \delta_{ij}$ for $i, j \in \mathcal{M}$. These functions satisfy the two identities

$$e_i e_j = \delta_{ij} e_j, \quad \sum_i e_i = \mathbb{I}. \quad (7)$$

As a consequence of the identities (7) and the Leibniz rule, the differentials de_i of a differential calculus on \mathcal{A} are subject to the following relations,

$$de_i e_j = -e_i de_j + \delta_{ij} de_j, \quad \sum_i de_i = 0. \quad (8)$$

Without additional constraints, we are dealing with the universal differential calculus $(\tilde{\Omega}(\mathcal{A}), \tilde{d})$. Introducing the 1-forms

$$e_{ij} = e_i \tilde{d}e_j \quad (i \neq j) \quad (9)$$

one finds that they form a basis over \mathbb{C} of the space $\tilde{\Omega}^1$ of universal 1-forms. Moreover, all first order differential calculi are obtained from the universal one by setting some of the e_{ij} to zero.

Let us associate with each nonvanishing e_{ij} of some differential calculus (Ω, d) an arrow from the point i to the point j :

$$e_{ij} \neq 0 \iff i \bullet \longrightarrow \bullet j. \quad (10)$$

The universal (first order) differential calculus then corresponds to the complete digraph where all vertices are connected with each other by a pair of antiparallel arrows. Other first order differential calculi are obtained by deleting some of the arrows. The choice of a (first order) differential calculus on a finite set therefore means assigning a connection structure to it. The latter is mirrored in the following formula for the differential of a function $f \in \mathcal{A}$,⁴

$$df = \sum_{i,j} [f(j) - f(i)] e_{ij}. \quad (11)$$

⁴More precisely, the summation runs over all i, j with $i \neq j$. Note that e_{ii} has not been defined. We may, however, set $e_{ii} := 0$.

Returning to the universal calculus, concatenation of the 1-forms e_{ij} leads to the *basic* $(r-1)$ -forms

$$e_{i_1 \dots i_r} := e_{i_1 i_2} e_{i_2 i_3} \cdots e_{i_{r-1} i_r} \quad (r > 1). \quad (12)$$

They constitute a basis of $\tilde{\Omega}^{r-1}$ over \mathbb{C} and satisfy the simple relations

$$e_{i_1 \dots i_r} e_{j_1 \dots j_s} = \delta_{i_r j_1} e_{i_1 \dots i_{r-1} j_1 \dots j_s}. \quad (13)$$

Furthermore, we have

$$\tilde{d}e_i = \sum_j (e_{ji} - e_{ij}) \quad (14)$$

$$\tilde{d}e_{ij} = \sum_k (e_{kij} - e_{ikj} + e_{ijk}) \quad (15)$$

$$\tilde{d}e_{ijk} = \sum_l (e_{lijk} - e_{iljk} + e_{ijlk} - e_{ijkl}) \quad (16)$$

\vdots

The first equation is a special case of (11). In a ‘reduced’ differential calculus (Ω, d) where not all of the e_{ij} are present, the possibilities to build (nonvanishing) higher forms $e_{i_1 \dots i_r}$ are restricted and the above formulas for $\tilde{d}e_{i_1 \dots i_r}$ impose further constraints on them.

Example. The graph drawn in Fig. 1 determines a first order differential calculus with nonvanishing basic 1-forms $e_{12}, e_{23}, e_{14}, e_{43}$. Concatenation only leads to e_{123} and e_{143} as possible nonvanishing basic 2-forms. There are no nonvanishing r -forms with $r > 2$.

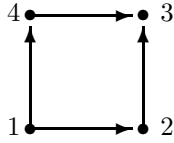


Fig. 1

The digraph associated with a special differential calculus on a set of four elements.

The graph is obtained from the complete digraph by deletion of some arrows. In particular, an arrow from point 1 to point 3 is missing which corresponds to setting e_{13} to zero in the universal differential calculus. This leads to $0 = de_{13} = -e_{123} - e_{143}$. ■

A discrete set together with a differential calculus on it is called a *discrete differential manifold*. [8]

3.1 Representations of first order differential calculi on finite sets

As explained above, first order differential calculi on a set of N elements are in bijective correspondence with digraphs with N vertices and at most a pair of antiparallel arrows between any two vertices. On the other hand, in graph theory such a digraph is characterized by its *adjacency matrix* which is an $N \times N$ -matrix \mathcal{D} such that $\mathcal{D}_{ij} = 1$ if there is an arrow from i to j and $\mathcal{D}_{ij} = 0$ otherwise. One should then expect that the (first order) differential calculus determined by a digraph can be expressed in terms of \mathcal{D} . The simplest way to build a derivation $d : \mathcal{A} \rightarrow \Omega^1(\mathcal{A})$ is as a commutator,

$$df := [\mathcal{D}, f] \quad (17)$$

which presumes, however, that the elements of \mathcal{A} can be represented as $N \times N$ -matrices. But this is naturally achieved via

$$f \mapsto \begin{pmatrix} f(1) & & 0 \\ & \ddots & \\ 0 & & f(N) \end{pmatrix}. \quad (18)$$

Comparison of (17) with our formula (11) shows that the basic 1-form e_{ij} is represented as the $N \times N$ -matrix E_{ij} with a 1 in the i th row and j th column and zeros elsewhere. The adjacency matrix \mathcal{D} represents $\sum_{i,j} e_{ij}$.

Proceeding beyond 1-forms, the above representation will not respect the \mathbb{Z}_2 -grading of a differential algebra $\Omega(\mathcal{A})$. One may consider instead a ‘doubled’ representation [9]

$$e_i \mapsto \begin{pmatrix} E_{ii} & 0 \\ 0 & E_{ii} \end{pmatrix}, \quad e_{ij} \mapsto \begin{pmatrix} 0 & E_{ij}^\dagger \\ E_{ij} & 0 \end{pmatrix}. \quad (19)$$

The grading can be expressed in terms of a grading operator which in our case is given by

$$\gamma := \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}. \quad (20)$$

It is selfadjoint and satisfies

$$\gamma^2 = \mathbf{1}, \quad \gamma \hat{\mathcal{D}} = -\hat{\mathcal{D}} \gamma \quad \gamma \hat{f} = \hat{f} \gamma \quad (21)$$

with

$$\hat{\mathcal{D}} := \begin{pmatrix} 0 & \mathcal{D}^\dagger \\ \mathcal{D} & 0 \end{pmatrix} \quad \hat{f} := \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} \quad (22)$$

where f has to be represented as in (18). In this way we do *not* in general obtain a representation of the first order differential calculus which we started with, however, but a representation of the corresponding ‘symmetric’ differential calculus where with $e_{ij} \neq 0$ there is also $e_{ji} \neq 0$ (so that the associated digraph is symmetric).

With the above representations of (first order) differential calculi we have established contact with Alain Connes’ formalism [10] of noncommutative geometry. But in the present context \mathcal{D} is *not*, in general, a *selfadjoint* operator (on the Hilbert space \mathbb{C}^N). The ‘doubling’ in (19) leads to a selfadjoint operator on the Hilbert space $\mathcal{H} = \mathbb{C}^{2N}$, however. $(\mathcal{A}, \mathcal{H}, \hat{\mathcal{D}})$ is an example of an *even spectral triple*, a central structure in Connes’ approach to noncommutative geometry.[10, 11] A *spectral triple* $(\mathcal{A}, \mathcal{H}, \hat{\mathcal{D}})$ consists of an involutive algebra \mathcal{A} of operators on a Hilbert space \mathcal{H} together with a selfadjoint operator $\hat{\mathcal{D}}$ satisfying some technical conditions. It is called *even* when there is a grading operator γ , as in our example.

4 Lattice differential calculus

Let $\mathcal{M} = \mathbb{Z}$. For $i, j \in \mathcal{M}$ we define a differential calculus by $e_{ij} \neq 0 \iff j = i+1$ following the rules described for finite sets in the previous section. This corresponds to the oriented linear lattice graph drawn in Fig. 2.



Fig. 2

An oriented linear lattice graph.

In this example we are dealing with an *infinite* set and thus infinite sums in some formulas which would actually require a bit more care. Introducing the lattice coordinate function

$$x := \ell \sum_j j e_j \quad (23)$$

with a real constant $\ell > 0$, one obtains

$$dx = \ell \sum_i i de_i = \ell \sum_{i,j} i (e_{ji} - e_{ij}) = \ell \sum_i i (e_{i-1,i} - e_{i,i+1}) = \ell \sum_i e_{i,i+1} \quad (24)$$

and

$$[dx, x] = \ell^2 \sum_{i,j} j [e_{i,i+1}, e_j] = \ell^2 \sum_i e_{i,i+1} = \ell dx \quad (25)$$

using (13). Hence

$$[dx, x] = \ell dx . \quad (26)$$

In the limit $\ell \rightarrow 0$ the lattice coordinate function x naively approximates the corresponding coordinate function on the real line \mathbb{R} . From our last equation we then recover the familiar commutativity of ordinary differentials and functions. The above commutation relation makes also sense, however, on \mathbb{R} when $\ell > 0$. We then have a *deformation* of the ordinary calculus of differential forms on \mathbb{R} with deformation parameter ℓ . In the following we collect some properties of this deformed differential calculus. Written in the form

$$dx x = (x + \ell) dx , \quad (27)$$

the above commutation relation extends to the algebra \mathcal{A} of all functions on \mathbb{R} as

$$dx f(x) = f(x + \ell) dx . \quad (28)$$

Furthermore,

$$\begin{aligned} df &= (\partial_{+x} f) dx = \frac{1}{\ell} (\partial_{+x} f) [dx, x] = \frac{1}{\ell} [(\partial_{+x} f) dx, x] \\ &= \frac{1}{\ell} [df, x] = \frac{1}{\ell} (d(fx - xf) - [f, dx]) = \frac{1}{\ell} (dx f - f dx) \\ &= \frac{1}{\ell} [f(x + \ell) - f(x)] dx \end{aligned} \quad (29)$$

so that the *left partial derivative* defined via the first equality turns out to be the right discrete derivative, i.e.,

$$\partial_{+x} f = \frac{1}{\ell} [f(x + \ell) - f(x)] . \quad (30)$$

Introducing a *right partial derivative* via $df = dx \partial_{-x} f$, an application of (28) shows that it is the left discrete derivative, i.e.,

$$\partial_{-x} f = \frac{1}{\ell} [f(x) - f(x - \ell)] . \quad (31)$$

An *indefinite integral* should have the property

$$\int df = f + \text{'constant'} \quad (32)$$

where ‘constants’ are functions annihilated by d . These are just the functions with period ℓ (so that $f(x + \ell) = f(x)$). It turns out that every function can be integrated.[12] Since the indefinite integral is only determined up to the addition of an arbitrary function with period ℓ , it defines a *definite integral* only if the region of integration is an interval the length of which is a multiple of ℓ (or a union of such intervals). Then one obtains

$$\int_{x_0 - m\ell}^{x_0 + n\ell} f(x) dx = \ell \sum_{k=-m}^{n-1} f(x_0 + k\ell) \quad (33)$$

and in particular

$$\int_{x_0 - \infty\ell}^{x_0 + \infty\ell} f(x) dx = \ell \sum_{k=-\infty}^{\infty} f(x_0 + k\ell) . \quad (34)$$

The integral simply picks out the values of f on a *lattice* with spacings ℓ and forms the Riemann integral for the corresponding piecewise constant function on \mathbb{R} . The point $x_0 \in \mathbb{R}$ determines how the lattice is embedded in \mathbb{R} .

Let now $\mathcal{M} = \mathbb{Z}^n$. For $a, b \in \mathcal{M}$ we define a differential calculus by

$$e_{ab} \neq 0 \iff b = a + \hat{\mu} \quad \text{where} \quad \hat{\mu} = (\delta_{\mu}^{\nu}) . \quad (35)$$

This corresponds to the oriented lattice graph drawn in Fig. 3.

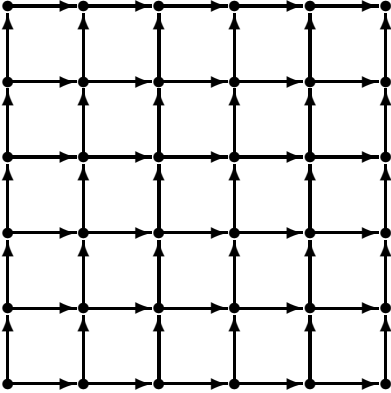


Fig. 3
A finite part
of the oriented
lattice graph.

Introducing the lattice coordinate functions

$$x^{\mu} := \ell^{\mu} \sum_a a^{\mu} e_a \quad (36)$$

with constants $\ell^{\mu} > 0$, one obtains

$$[dx^{\mu}, x^{\nu}] = \ell^{\mu} \delta^{\mu\nu} dx^{\mu} \quad (37)$$

using (13). $\{dx^{\mu}\}$ is a left \mathcal{A} -module basis of Ω^1 . The above commutation relations define a deformation of the ordinary differential calculus on \mathbb{R}^n . This deformed differential calculus may be regarded as a basic structure underlying lattice field theories.[13]

5 A class of noncommutative differential calculi on a commutative algebra

Let \mathcal{A} be the associative and commutative algebra over \mathbb{R} freely generated by elements x^μ , $\mu = 1, \dots, n$. For example, the x^μ could be the canonical coordinates on \mathbb{R}^n . The ordinary differential calculus on \mathcal{A} has the property $[dx^\mu, x^\nu] = 0$, i.e., differentials and functions commute. Relaxing this property, there is a class of *noncommutative differential calculi* such that⁵

$$[dx^\mu, x^\nu] = C^{\mu\nu}{}_\kappa dx^\kappa \quad (38)$$

with *structure functions* $C^{\mu\nu}{}_\kappa(x^\lambda)$ which have to satisfy some consistency conditions. First, we have

$$\begin{aligned} [dx^\mu, x^\nu] &= (dx^\mu) x^\nu - x^\nu dx^\mu \\ &= d(x^\mu x^\nu - x^\nu x^\mu) - x^\mu dx^\nu + (dx^\nu) x^\mu = [dx^\nu, x^\mu]. \end{aligned} \quad (39)$$

Assuming the differentials dx^μ , $\mu = 1, \dots, n$, to be linearly independent⁶, this implies

$$C^{\mu\nu}{}_\kappa = C^{\nu\mu}{}_\kappa. \quad (40)$$

Furthermore,

$$\begin{aligned} 0 &= ([dx^\mu, x^\nu] - C^{\mu\nu}{}_\kappa dx^\kappa) x^\lambda = [(dx^\mu) x^\lambda, x^\nu] - C^{\mu\nu}{}_\kappa (dx^\kappa) x^\lambda \\ &= [x^\lambda dx^\mu + C^{\mu\lambda}{}_\rho dx^\rho, x^\nu] - C^{\mu\nu}{}_\kappa (x^\lambda dx^\kappa + C^{\kappa\lambda}{}_\rho dx^\rho) \\ &= x^\lambda [dx^\mu, x^\nu] + C^{\mu\lambda}{}_\rho [dx^\rho, x^\nu] - x^\lambda C^{\mu\nu}{}_\kappa dx^\kappa - C^{\mu\nu}{}_\kappa C^{\kappa\lambda}{}_\rho dx^\rho \\ &= (C^{\mu\lambda}{}_\rho C^{\rho\nu}{}_\kappa - C^{\mu\nu}{}_\rho C^{\rho\lambda}{}_\kappa) dx^\kappa \end{aligned} \quad (41)$$

which leads to

$$C^{\lambda\mu}{}_\rho C^{\nu\rho}{}_\kappa = C^{\nu\mu}{}_\rho C^{\lambda\rho}{}_\kappa \quad (42)$$

or, in terms of the matrices C^μ with entries $(C^\mu)^\nu{}_\kappa := C^{\mu\nu}{}_\kappa$,

$$C^\mu C^\nu = C^\nu C^\mu. \quad (43)$$

For constant $C^{\mu\nu}{}_\kappa$ and $n \leq 3$, a classification of all solutions of the consistency conditions (40) and (42) has been obtained.[13, 14] Besides the ‘lattice differential calculus’ discussed in the previous subsection, this includes other interesting deformations of the ordinary differential calculus on \mathbb{R}^n . [15, 16] The relations (37) are obviously not invariant under (suitable) coordinate transformations. Invariance is achieved, however, with the form (38) of the commutation relations.

From the structure functions we can build

$$g^{\mu\nu} := \text{Tr}(C^\mu C^\nu) \quad (44)$$

which for the lattice calculus (37) becomes $(\ell^\mu)^2 \delta^{\mu\nu}$, a kind of metric tensor. In the framework under consideration, the metric arises as a composed object. The set of structure functions $C^{\mu\nu}{}_\kappa$ is the more fundamental geometric structure.

⁵On the rhs of this equation we are using the summation convention.

⁶More precisely, we assume here that the dx^μ form a left \mathcal{A} -module basis of $\Omega^1(\mathcal{A})$.

6 An application in the context of integrable models

For two-dimensional σ -models there is a construction of an infinite sequence of conserved currents [5] which can be formulated very compactly in terms of ordinary differential forms. This then suggests to generalize the notion of a σ -model to noncommutative differential calculi such that the construction of conservation laws still works. In this way one obtains a simple though very much non-trivial application of the formalism of noncommutative geometry.[6, 7]

6.1 Generalized integrable σ -models

Let \mathcal{A} be an associative and commutative⁷ algebra with unit $\mathbb{1}$ and (Ω, d) a differential calculus on it. Furthermore, let $\star : \Omega^1 \rightarrow \Omega^1$ be an invertible linear map such that

$$\star(wf) = f \star w \quad (45)$$

and

$$w \star w' = w' \star w. \quad (46)$$

In addition, we require that

$$dw = 0 \quad \Rightarrow \quad w = \star \star d\chi \quad (47)$$

with $\chi \in \mathcal{A}$. Furthermore, let $a \in GL(n, \mathcal{A})$ and $A := a^{-1} da$. Then

$$F := dA + AA \equiv 0 \quad (48)$$

since $da^{-1} = -a^{-1}(da)a^{-1}$. These definitions are made in such a way that the field equation of a *generalized σ -model*

$$d \star A = 0 \quad (49)$$

and a construction of an infinite set of conservation laws in two dimensions[5] generalizes to a considerably more general framework. Let χ be an $n \times n$ matrix with entries in \mathcal{A} . Using the two relations (45), (46), and the field equation $d \star A = 0$, we find

$$d \star (A^i_j \chi^j_k) = d(\chi^j_k \star A^i_j) = (d\chi^j_k) \star A^i_j + \chi^j_k d \star A^i_j = A^i_j \star d\chi^j_k \quad (50)$$

and thus

$$d \star D\chi = d \star d\chi + d(\star A\chi) = d \star d\chi + A \star d\chi = D \star d\chi \quad (51)$$

where $D\chi := d\chi + A\chi$. Let

$$\chi^{(0)} := \begin{pmatrix} \mathbb{1} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathbb{1} \end{pmatrix} \quad (52)$$

Then

$$J^{(1)} := D\chi^{(0)} = A \quad (53)$$

so that

$$d \star J^{(1)} = 0 \quad (54)$$

⁷A generalization of the following to *noncommutative* algebras seems to be possible if they admit an involution † . Then (45) has to be replaced by $\star(wf) = f^\dagger \star w$.

as a consequence of the field equation. Hence, using (47),

$$J^{(1)} = \star d\chi^{(1)} \quad (55)$$

with a matrix $\chi^{(1)}$. Now, let $J^{(m)}$ be a conserved current, i.e.,

$$J^{(m)} = \star d\chi^{(m)} . \quad (56)$$

Then

$$J^{(m+1)} := D\chi^{(m)} \quad (m \geq 0) \quad (57)$$

is also conserved since

$$\begin{aligned} d \star J^{(m+1)} &= d \star D\chi^{(m)} = D \star d\chi^{(m)} = DJ^{(m)} = D^2\chi^{(m-1)} \\ &= F\chi^{(m-1)} = 0 \quad (m \geq 1) \end{aligned} \quad (58)$$

using (51) and the identity (48). Starting with $J^{(1)}$, we obtain an infinite set of (matrices of) conserved currents.⁸ In case of the ordinary differential calculus on a two-dimensional Riemannian manifold, this construction reduces to the classical one.[5]

Let us (formally) define

$$\chi := \sum_{m \geq 0} \lambda^m \chi^{(m)} \quad (59)$$

with a constant $\lambda \neq 0$. Then (56) and (57) lead to

$$\star d\chi = \lambda D\chi . \quad (60)$$

As a consequence of this equation we get

$$0 = d \star D\chi^i_j = D \star d\chi^i_j + \chi^k_j d \star A^i_k \quad (61)$$

and

$$D \star d\chi = \lambda D^2\chi = \lambda F\chi . \quad (62)$$

Using $A = a^{-1} da$, the integrability condition of the linear equation (60) is the field equation (49).

We have extended the definition of a class of σ -models to a rather general framework of noncommutative geometry, though still with the restriction to a commutative algebra \mathcal{A} (which can always be realized as an algebra of functions on some space), but with *noncommutative* differential calculi (where functions and differentials do not commute, in general). Already in this case a huge set of possibilities for integrable models appears.[6, 7]

6.2 Example: recovering the Toda lattice

Let \mathcal{A} be the (commutative) algebra of functions on $\mathcal{M} = \ell_0\mathbb{Z} \times \ell_1\mathbb{Z}$. Here $\ell_k\mathbb{Z}$ stands for the one-dimensional lattice with spacings $\ell_k > 0$. A special differential calculus $(\Omega(\mathcal{A}), d)$ on \mathcal{A} is then determined by the following commutation relations,

$$[dt, t] = \ell_0 dt, \quad [dx, x] = \ell_1 dx, \quad [dt, x] = [dx, t] = 0 \quad (63)$$

⁸There is no guarantee, however, that all these currents are really independent. For example, our formalism includes the free linear wave equation on two-dimensional Minkowski space. In that case, the higher conserved charges are just polynomials in the first one (which is the total momentum).

where t and x are the canonical coordinate functions on $\ell_0\mathbb{Z}$ and $\ell_1\mathbb{Z}$, respectively. This is our lattice differential calculus (37) for $n = 2$. As a consequence, we have

$$dt f(\mathbf{x}) = f(\mathbf{x} + \ell_0) dt, \quad dx f(\mathbf{x}) = f(\mathbf{x} + \ell_1) dx \quad (64)$$

where

$$\mathbf{x} := (t, x), \quad \mathbf{x} + \ell_0 := (t + \ell_0, x), \quad \mathbf{x} + \ell_1 := (t, x + \ell_1). \quad (65)$$

Furthermore,

$$df = \frac{1}{\ell_0} \{f(\mathbf{x} + \ell_0) - f(\mathbf{x})\} dt + \frac{1}{\ell_1} \{f(\mathbf{x} + \ell_1) - f(\mathbf{x})\} dx. \quad (66)$$

Acting with d on (63), we obtain

$$dt dx = -dx dt, \quad dt dt = 0 = dx dx. \quad (67)$$

This familiar anticommutativity of differentials does not extend to general 1-forms, however. The differential calculus has the following property.

Lemma. Every closed 1-form is exact.

Proof: For $w = w_0(t, x) dt + w_1(t, x) dx$ the condition $dw = 0$ means $\partial_{+t} w_1 = \partial_{+x} w_0$. For simplicity, we set $\ell_0 = \ell_1 = 1$ in the following. Let us define⁹

$$F(t, x) := \sum_{k=0}^{t-1} w_0(k, 0) + \sum_{j=0}^{x-1} w_1(t, j).$$

It satisfies

$$\partial_{+x} F = F(t, x+1) - F(t, x) = \sum_{j=0}^x w_1(t, j) - \sum_{j=0}^{x-1} w_1(t, j) = w_1(t, x)$$

and, using $dw = 0$,

$$\begin{aligned} \partial_{+t} F &= F(t+1, x) - F(t, x) = w_0(t, 0) + \sum_{j=0}^{x-1} \partial_{+t} w_1(t, j) \\ &= w_0(t, 0) + \sum_{j=0}^{x-1} \partial_{+x} w_0(t, j) = w_0(t, 0) + w_0(t, x) - w_0(t, 0) = w_0(t, x). \end{aligned}$$

Hence $w = dF$. ■

Let us now turn to the conditions for the \star -operator. First we introduce $g^{\mu\nu}$ via

$$dx^\mu \star dx^\nu = g^{\mu\nu} dt dx. \quad (68)$$

With $w = w_\mu dx^\mu$, (46) becomes

$$[w_\mu(\mathbf{x}) w'_\nu(\mathbf{x} + \ell_\mu - \ell_\nu) - w'_\mu(\mathbf{x}) w_\nu(\mathbf{x} + \ell_\mu - \ell_\nu)] g^{\mu\nu} = 0 \quad \forall w_\mu, w'_\mu. \quad (69)$$

It yields

$$g^{\mu\nu} = c^\mu \delta^{\mu\nu} \quad (70)$$

⁹This function is obtained by integrating w along a path $\gamma : \mathbb{N} \rightarrow \mathbb{Z}^2$ first from $(0, 0)$ to $(t, 0)$ along the t -lattice direction, then from $(t, 0)$ to (t, x) along the x -lattice direction. The result does not depend on the chosen path. This follows from an application of Stokes' theorem.

with arbitrary functions $c^\mu(\mathbf{x})$ which have to be different from zero in order for \star to be invertible. This includes the metric (44). For the generalized Hodge operator we now obtain

$$\star dt = c^0(\mathbf{x} - \ell_0) dx, \quad \star dx = -c^1(\mathbf{x} - \ell_1) dt \quad (71)$$

which extends to Ω^1 via (45).

In the following, we choose $g^{\mu\nu} = \eta^{\mu\nu}$ which in classical differential geometry are the components of the two-dimensional Minkowski metric with respect to an inertial coordinate system. We then have $\star \star w(\mathbf{x}) = w(\mathbf{x} - \ell_0 - \ell_1)$ which, together with the above Lemma, implies (47). Therefore, the construction of conservation laws does work in the case under consideration. Let us look at the simplest generalized σ -model where a is just a function (i.e., a 1×1 -matrix). We write

$$a = e^{-q(t,x)} \quad (72)$$

with a function q and $q_k(n) := q(n\ell_0, k\ell_1)$. Then

$$A = \frac{1}{\ell_0}(e^{q_k(n)-q_k(n+1)} - 1) dt + \frac{1}{\ell_1}(e^{q_k(n)-q_{k+1}(n)} - 1) dx \quad (73)$$

$$\star A = -\frac{1}{\ell_0}(e^{q_k(n-1)-q_k(n)} - 1) dx - \frac{1}{\ell_1}(e^{q_{k-1}(n)-q_k(n)} - 1) dt \quad (74)$$

and the field equation $d \star A = 0$ takes the form

$$\frac{1}{\ell_0^2} [e^{q_k(n-1)-q_k(n)} - e^{q_k(n)-q_k(n+1)}] = \frac{1}{\ell_1^2} [e^{q_{k-1}(n)-q_k(n)} - e^{q_k(n)-q_{k+1}(n)}] \quad (75)$$

Replacing \mathcal{A} with the algebra of functions on $\mathbb{R} \times \ell_1 \mathbb{Z}$ which are smooth in the first argument, the limit $\ell_0 \rightarrow 0$ can be performed. This contraction leads to

$$\ddot{q}_k + \frac{1}{\ell_1^2} (e^{q_k - q_{k+1}} - e^{q_{k-1} - q_k}) = 0 \quad (76)$$

which is the *nonlinear Toda lattice* equation [17]. In particular, in this way a new and simple understanding of its complete integrability has been achieved. There is a ‘noncommutative geometry’ naturally associated with the Toda lattice equation. Generalizations of the Toda lattice are obtained by replacing the function a with a $GL(n, \mathcal{A})$ -matrix.[6]

7 Metrics in noncommutative geometry

In the previous section we have introduced a generalized Hodge \star -operator. In classical (Riemannian) differential geometry, the Hodge operator contains information equivalent to a metric tensor which in turn has its origin in the problem of defining the length of a curve and the distance between points of a Riemannian space. On the basis of the formalism sketched in section 3.1, Connes proposed a generalization of the classical distance function to discrete and even noncommutative spaces.[4] Some examples are discussed in the following subsection.[19] The relation with a generalized Hodge operator or other generalized concepts of a metric still has to be understood, however.

7.1 Connes’ distance function associated with differential calculi on finite sets

Let $(\mathcal{A}, \mathcal{H}, \hat{\mathcal{D}})$ be a spectral triple (cf section 3.1). A *state* on \mathcal{A} is a linear map $\phi : \mathcal{A} \rightarrow \mathbb{C}$ which is positiv, i.e., $\phi(a^*a) \geq 0$ for all $a \in \mathcal{A}$, and normalized, i.e., $\phi(\mathbb{1}) = 1$. According to Connes [4], the distance between two states ϕ and ϕ' is given by

$$d(\phi, \phi') := \sup\{|\phi(a) - \phi'(a)| ; a \in \mathcal{A}, \|[\hat{\mathcal{D}}, a]\| \leq 1\}. \quad (77)$$

Given a set \mathcal{M} , each point $p \in \mathcal{M}$ defines a state ϕ_p via $\phi_p(f) := f(p)$ for all functions f on \mathcal{M} . The above formula then becomes

$$d(p, p') := \sup\{|f(p) - f(p')| ; f \in \mathcal{A}, \|[\hat{\mathcal{D}}, f]\| \leq 1\} . \quad (78)$$

Example 1. The universal first order differential calculus on a set of two elements p, q is described by a graph consisting of two points which are connected by a pair of antiparallel arrows. Its adjacency matrix is

$$\mathcal{D} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (79)$$

so that

$$[\mathcal{D}, f] = \begin{pmatrix} 0 & f(p) - f(q) \\ f(q) - f(p) & 0 \end{pmatrix} . \quad (80)$$

Then

$$\begin{aligned} \|[\mathcal{D}, f]\|^2 &= \sup_{\|\psi\|=1} \|[\mathcal{D}, f] \psi\|^2 = \sup_{\|\psi\|=1} |f(p) - f(q)|^2 (|\psi_1|^2 + |\psi_2|^2) \\ &= |f(p) - f(q)|^2 \end{aligned} \quad (81)$$

for $\psi \in \mathbb{C}^2$. It follows that Connes' distance function defined with the adjacency matrix gives $d(p, q) = 1$. In this example, which appeared in models of elementary particle physics [18], there is no need for a 'doubling' of the representation as in (22). We may, however, replace \mathcal{D} by $\hat{\mathcal{D}}$ also in this case. The result for the distance between the two points remains unchanged, however.

Example 2. [19] Let us consider the first order differential calculus on a set of N elements determined by the graph in Fig. 4.



Fig. 4

A finite oriented linear lattice graph.

The corresponding adjacency matrix is

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & & 1 \\ 0 & \cdots & \cdots & & 0 \end{pmatrix} . \quad (82)$$

This matrix contains all the topological information about the lattice, i.e., the neighbourhood relationships. We can add information about the distances between neighbouring points to it in the following way. Let ℓ_k be the distance from point k to point $k + 1$ (numbering the lattice sites by $1, \dots, N$). We define

$$\mathcal{D}_N := \begin{pmatrix} 0 & \ell_1^{-1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & & \ell_{N-1}^{-1} \\ 0 & \cdots & \cdots & & 0 \end{pmatrix} . \quad (83)$$

With a complex function f we associate a real function F via

$$F_1 := 0, \quad F_{i+1} := F_i + |f_{i+1} - f_i| \quad i = 1, \dots, N-1 . \quad (84)$$

where $f_i := f(i)$. Then $|F_{i+1} - F_i| = |f_{i+1} - f_i|$ and

$$\|[\hat{\mathcal{D}}_N, \hat{f}] \psi\| = \|[\hat{\mathcal{D}}_N, \hat{F}] \psi\| \quad (85)$$

for all $\psi \in \mathbb{C}^{2N}$. Hence, in calculating the supremum over all functions f in the definition of Connes' distance function, it is sufficient to consider *real* functions. Then $Q_N := [\hat{\mathcal{D}}_N, \hat{f}]$ is anti-hermitean and its norm is then given by the maximal absolute value of its eigenvalues. Instead of Q_N it is simpler to consider $Q_N Q_N^\dagger$ which is already diagonal with entries $0, \ell_1^{-2}(f_2 - f_1)^2, \dots, \ell_{N-1}^{-2}(f_N - f_{N-1})^2, \ell_1^{-2}(f_2 - f_1)^2, \dots, \ell_{N-1}^{-2}(f_N - f_{N-1})^2, 0$ on the diagonal. This implies

$$\|[\hat{\mathcal{D}}_N, \hat{f}]\| = \max\{\ell_1^{-1}|f_2 - f_1|, \dots, \ell_{N-1}^{-1}|f_N - f_{N-1}|\}. \quad (86)$$

We have the obvious inequality

$$d(i, i+k) \leq \sup\{|f(i+1) - f(i)| + \dots + |f(i+k) - f(i+k-1)|; \|[\hat{\mathcal{D}}_N, \hat{f}]\| \leq 1\}. \quad (87)$$

But a closer inspection shows that actually equality holds here. We conclude that $d(i, i+k) = \ell_i + \ell_{i+1} + \dots + \ell_{i+k-1}$.¹⁰ ■

Example 3. The graph in Fig. 1 has the adjacency matrix

$$\mathcal{D} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (88)$$

The norm of $[\hat{\mathcal{D}}, \hat{f}]$ is the positive square root of the largest eigenvalue of

$$[\hat{\mathcal{D}}, \hat{f}] [\hat{\mathcal{D}}, \hat{f}]^\dagger = \begin{pmatrix} [\mathcal{D}, f^*]^\dagger [\mathcal{D}, f^*] & 0 \\ 0 & [\mathcal{D}, f] [\mathcal{D}, f]^\dagger \end{pmatrix}. \quad (89)$$

It follows that

$$\|[\hat{\mathcal{D}}, \hat{f}]\| = \max\{\sqrt{|f_{21}|^2 + |f_{41}|^2}, \sqrt{|f_{32}|^2 + |f_{34}|^2}\} \quad (90)$$

where $f_{kl} := f_k - f_l$. Introducing

$$x := \frac{1}{2}(f_{21} - f_{41}), \quad y := \frac{1}{2}(f_{21} + f_{41}), \quad z := \frac{1}{2}(f_{32} + f_{34}), \quad (91)$$

we find the 'Euclidean' result

$$d(1, 3) = \sup\{\frac{1}{2}|y + z|; \max\{|x|^2 + |y|^2, |x|^2 + |z|^2\} \leq 2\} = \sqrt{2}. \quad (92)$$

■

Example 4. Let us consider the following digraph which is part of the lattice graph in Fig. 3.

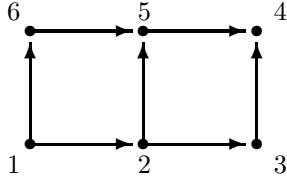


Fig. 5

The digraph associated with a special first order differential calculus on a set of six points.

¹⁰ A different choice of $\hat{\mathcal{D}}$ has been made elsewhere [20] to define the distance on a lattice. See also Rieffel [21] for a reformulation of discrete metric spaces in Connes' framework.

A numerical evaluation of the distance function shows that $d(3, 6) \lesssim 2 < d(1, 4) < \sqrt{5}$ and thus deviates from the Euclidean value. ■

The last example shows that Connes' distance function defined in terms of the adjacency matrix of the n -dimensional oriented lattice graph with $n > 1$ (see Fig. 3) does *not* assign to it a Euclidean geometry, as might have been conjectured on the basis of our examples 2 and 3.

8 Final remarks

This work centered around examples which live on lattices. Such spaces do not at all exhaust the possibilities of noncommutative geometry of commutative algebras. In this case, and more generally in the case of discrete spaces, the generalized partial derivatives of a differential calculus are *discrete* derivatives, corresponding to an infinite sum of powers of ordinary partial derivatives. There are other differential calculi where the generalized partial derivatives are differential operators of finite order and some of them appear to be of relevance for an analysis of soliton equations, for example.[22] There is much more to mention in this context and we refer to a recent review [2] for further information and a guide to the relevant literature.

An important aspect of the formalism of noncommutative geometry is a technical one. On the level of generalized differential forms we have very compact expressions which are easy to handle thanks to the simple rules of differential calculus. Decomposed into components, however, we end up with rather complicated formulas, in general. This is precisely the experience which especially relativists make when they encounter the Cartan formalism in general relativity. Our generalization of the construction of conserved currents for (generalized) σ -models reviewed in section 6.1 is another nice example.

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